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DUALITY IN HOMOTOPY THEORY: A RETROSPECTIVE ESSAY

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For Saunders MacLane, on the occasion of his 70th birthday

1.

The notions of *category, functor* and *natural transformation* were formally introduced into mathematics by Eilenberg and MacLane in 1942 [20], the detailed exposition appearing in 1945 [21]. A particularly important aspect of their work was that of the *duality* which is automatically present whenever categorical concepts are in question. In the ensuing years, MacLane was very much concerned to make such concepts part of the equipment of the working mathematician (see the title of his important work [41]). The concept of duality might then be put to work in the following way. Let Q be a statement (or a definition) which is meaningful in any category $\mathscr{C}$. We may make the statement Q in the dual category $\mathscr{C}^*$ and *interpret* the statement in $\mathscr{C}$; we thus obtain a new statement Q^* which we may call the dual of Q, and we may write

$$Q^*(\mathscr{C}) = Q(\mathscr{C}^*).$$

For example, we may define a *monomorphism* to be a morphism f such that $fu = fv \Rightarrow u = v$. A morphism f in $\mathscr{C}$ is a monomorphism in $\mathscr{C}^*$ if it has the property that, *in* $\mathscr{C}$, $uf = vf \Rightarrow u = v$. We thus obtain the definition dual to that of monomorphism; it is the definition of an *epimorphism*. Any statement about monomorphisms which is meaningful (true) in every category may be dualized to a statement about epimorphisms which is meaningful (true) in every category. Further examples of dual concepts are product and coproduct, initial and terminal objects, kernel and cokernel (since the notion of a zero morphism is self-dual).

Of course categorical statements and definitions may be specialized to particular categories. The converse process is, however, far more interesting – we may take a notion appropriate to a given category, say a notion of group theory or of topology, and then attempt to cast this notion in categorical language. Such a process may well have many different outcomes; but each successful attempt enables us to dualize the notion and then respecialize the dual notion in our original category of interest. We may loosely describe the resulting notion (of group theory or of topology) as dual to the original notion, so long as it is understood that this duality is not canonical and our choice of "dual", in this empirical sense, will be guided by considerations of usefulness.

MacLane [33,34] was the first to initiate such a program, particularly in the category of groups. In 1947 he was in Zürich and pointed out to Beno Eckmann that, in this sense, direct products of groups and free products of groups stood in a dual relation to each other. In making this observation, MacLane was, of course, giving expression to what is now a commonplace approach – that of definition by a universal mapping property. The work of Eckmann and myself on duality in homotopy theory was very much inspired by MacLane's pioneering work – we worked in the homotopy category of based topological spaces where he had been principally concerned with the category of groups. In both cases, since we were concerned with the more heuristic notion of duality *within* a given category, rather than with categorical duality, choices had to be made to determine appropriate dual concepts, and the truth of a dual assertion could not necessarily be inferred from the truth of the assertion, unless the proof of the assertion could be "lifted" into a general category, that is, could be expressed in universal terms rather than in terms meaningful and valid only in the special category under consideration. Just as MacLane had been interested in the late 1940's in pairs of dual assertions in group theory, both of which were both interesting and true, so, in the work of Eckmann and myself beginning in 1955, we were interested in such pairs of dual assertions in homotopy theory.

In this retrospective survey, I would like to describe how Eckmann and I arrived at our formulation of the duality and to indicate some of the developments due to others as well as ourselves. I would also like to take this opportunity to express my own personal indebtedness to both Saunders MacLane and Beno Eckmann. Indeed, the year 1947, already mentioned in connection with the influence of MacLane's ideas on Beno Eckmann, was a crucial year for me. For it was in 1947, when I was a research student of Henry Whitehead in Oxford, that I myself first met both Saunders MacLane and Beno Eckmann. Saunders infected me with his enormous enthusiasm for mathematics and his dynamic personality, while Beno gave me great encouragement by treating me already then as a mature mathematician – which I certainly was not. To both of them I am deeply grateful and much indebted.

The next three sections of this essay consist of a brief, and rather subjective, history of the development of duality in homotopy theory. Many names should have been mentioned, of those who worked in the field and others who influenced the thinking of Eckmann and myself. Justice to such friends and colleagues is only partially done in the text and in the bibliography which closes this essay.

2.

In 1955 I was invited to visit Warsaw and Zürich. In Warsaw I was the guest of Karol Borsuk; in Zürich of Beno Eckmann. Borsuk was at that time considering a notion he had introduced of the *dependence* of maps. Let $f : X \to Y$, $g : X \to Z$ be continuous maps (we assume here that all spaces and pairs are CW). Then we say that g depends on f if, for any pair $(\bar{X}, X)$ – that is, X is a subcomplex of $\bar{X}$ – the extendability of f to $\bar{X}$ implies the extendability of g. Borsuk[1] also said that f was a *multiple* of g, or that g was a *divisor* of f. It is interesting that Thom had independently introduced the idea of the dependence of cohomology classes (of a given space X) and that Thom's notion is subsumed under this notion of dependence by taking Y and Z to be Eilenberg–MacLane complexes. The following Theorem is essentially due to Borsuk and splendidly justifies his alternative terminology.

Theorem 2.1. *The map $g : X \to Z$ depends on the map $f : X \to Y$ if and only if there exists $h : Y \to Z$ with $g \simeq hf$.*

I would like to interpose here a rather significant reminiscence. Borsuk had asked the question whether, if two maps $f, g : X \to Y$ had the property that each depended on the other, it followed that $g \simeq hf$ for some homotpy equivalence $h : Y \to Y$. That this does not follow may be seen by considering the elements ω, $5\omega \in \pi_6(S^3)$, where ω is the canonical generator[2]. Since S^3 is a topological group, it follows that $5\omega = 5(\iota \circ \omega)$, so that 5ω depends on ω. On the other hand, since ω is of order 12, $\omega = 5\iota \circ 5\omega$, so that ω depends on 5ω. On the other hand, ω and 5ω are not connected by a homotopy self-equivalence of S^3, since such an equivalence would convert ω into $\pm\omega$ and $5\omega \neq \pm\omega$. I had communicated this counterexample to Borsuk, yet, a year later, a student of Borsuk's told Ganea, then visiting Warsaw, that he was still looking for a counterexample. "But did not Hilton provide a counterexample?" asked Ganea. "Yes, but his counterexample was algebraic" replied the young man!

To return to my narrative, I reached Zürich from Warsaw in September, 1955, and was soon working closely with Beno Eckmann, then very much interested in module theory. I informed Beno of my work in Warsaw on Borsuk dependence, and, in particular, of Theorem 2.1, and Beno suggested we should study the same question for Λ-modules. It soon became clear that we should expect the result analogous to Theorem 2.1 to hold, provided we could find a suitable definition of homotopy for module-maps. Since such maps may be subtracted, it suffices to define nullhomotopy and the analogy then suggests that a module-map $\varphi : A \to B$ is *nullhomotopic* if it extends to every supermodule of A. Plainly this is equivalent to requiring that φ factor through an injective module or that it extend to some injective supermodule of A. We decided to call such an injective supermodule of A a *cone* on A and to write it CA. It then seemed reasonable to call the quotient CA/A a *suspension* of A and write it ΣA. What was new in this was not the construction of

[1] Actually, Borsuk only considered the case $Y = Z$; but our generalization here is both natural and convenient and was adopted, e.g., in [38].

[2] No homotopy theorist can afford not to be intimate with $\pi_6(S^3)$.

cones and suspensions in this sense – these were familiar constructions of module theory since injective resolutions had been studied – but the point of view that they played a role analogous to that of cones and suspensions in homotopy theory.

Much of the apparatus of homotopy theory could now be reproduced for modules. With our notion of module homotopy (which we soon decided to call *injective homotopy*), the homotopy extension property is universally verified. We can form track groups and mapping cylinders; we can relativize; we can define fibre maps (by homotopy lifting) and obtain the usual theorem setting up an isomorphism between the relative groups of total space modulo fibre and the absolute groups of the base. There are the usual exact sequences. Further, as we have said, the analogue of Theorem 2.1 holds.

Now the duality in module theory suggested that we should also introduce *projective homotopy*. We did not, at that stage, argue, as we might have done, within an abelian category with sufficient injectives (or projectives) but it was fairly automatic to obtain duals of all our previous results; of course, a module-map $\varphi : A \to B$ is declared to be *projectively nullhomotopic* if it may be lifted into every module having B as a quotient, and this is equivalent to demanding that φ factor through some projective module, or that it lift to some projective module having B as a quotient. Then we obtain (contractible) path spaces, loop spaces, and, as before, so much of the apparatus of ordinary homotopy theory.

The next step was decisive for us; it was the observation that *both* injective homotopy and projective homotopy were the analogues of ordinary homotopy theory. For, in projective homotopy theory, every surjection is a fibre-map; and in the homotopy theory of spaces, a map $f : X \to Y$ is nullhomotopic if and only if it lifts to the total space of every fibre-map over Y. Thus the duality between injective homotopy and projective homotopy in the theory of Λ-modules translated back into an *internal duality* in the homotopy theory of topological spaces.

In the years 1955–57, Beno Eckmann and I presented our ideas on homotopy theory and duality at various conferences. We early recognized the connection between the internal duality and Kan's important theory of adjoint functors [31]. Then in 1958 we wrote a series of notes in the Comptes Rendus [1–4] and in 1959 a further note appeared [6], all elaborating the duality in the homotopy theory of spaces, giving some new results but also setting known results into the framework of this sometimes heuristic, sometimes categorical duality. We should emphasize that we were not, at that time, adopting a strictly categorical approach; we were working firmly within the category of based topological spaces and based maps. Our first three notes had the general title *Groupes d'homotopie et dualité,* indicating that the duality was intended to clarify the study of (generalized) homotopy groups; the study of homotopy groups was not intended merely to exemplify the duality[3]. We

[3] It was this emphasis which led us, finally, to regard homotopy groups with coefficients as sets of homotopy classes of maps of Moore spaces, rather than (as Peterson suggested) of 'co-Moore' spaces. We knew that the duality favored the latter; but, since Moore spaces always exist, we felt that the interests of homotopy theory favored the former!

did not yet use the "co" terminology at all systematically, often preferring to use a prime superscript. (One of our reasons for this was that this was *not* the duality in which homology and cohomology are dual concepts.) Thus our first note [1] dealt with H-structures and H'-structures on spaces. An H-structure on Y is a map $Y\times Y\rightarrow Y$ satisfying certain axioms, while an H'-structure on X is a map $X\rightarrow X\vee X$ satisfying the dual axioms. An H-structure on Y induces an H-structure on the set $[X, Y]$ of homotopy classes of maps of X into Y which is natural in X; the converse also holds. So does the dual: an H'-structure on X induces an H-structure on $[X, Y]$ which is natural in Y, and the converse holds. In the first case we have a generalization of the cohomology groups of X (obtained by taking Y to be an Eilenberg–MacLane space), in the second case we have a generalization of the homotopy groups of Y (obtained by taking X to be a sphere). A useful result is that an H-structure on Y and an H'-structure on X induce the *same* H-structure on $[X, Y]$; this structure on $[X, Y]$ must therefore be independent of the given structures on X and Y, and it must be commutative. This last assertion generalizes the classical result that the fundamental group of a topological group is commutative.

Among the H-spaces are the loop-spaces, which are groups in the homotopy category, and among the H'-spaces are the suspensions, which are cogroups in the homotopy category. There are, of course, H-spaces which are not groups – the best known example is S^7. The first example of an H'-space which is not a cogroup was given in [14]; it is the space $S^3\cup_{4\omega}e^7$(it would have sufficed to take $S^3\cup_{2\omega}e^7$). It was shown in [14] that this space is an H'-space but not a suspension and Berstein later showed that this space could support no homotopy-associative H'-structure. It is still unknown whether there are (homotopy) cogroups which are not suspensions, but one must conjecture that there are. For any integer $n\geq 0$ we may introduce the nth homotopy group $\pi_n(X;\ Y)$ of maps of X into Y and this construction is self-dual and simultaneously generalizes the cohomology groups (of X) and the homotopy groups (of Y). We may relativize on either variable and thus obtain dual exact sequences. This was described in [2], where there occurred, for the first time to our knowledge, the occurrence of the term 'cofibration' to describe the inclusion of a subspace in a space, where the pair has the homotopy extension property[4].

The duality itself suggested a notion dual to that of an *operator,* and this was exploited in [7]. A *cooperation* typically occurs in the following situation, extensively studied by Puppe [35]. Let $f: A\rightarrow V$ be a map and let W be the mapping cone of f. Thus W is obtained from the disjoint union of the (reduced) cone CA and V by means of the identification $(a,1)=fa,\ a\in A$. There is then a map $s: W\rightarrow W\vee\Sigma A$, given by

$$s(a,t)=(a,2t),\qquad 0\leq t\leq \tfrac{1}{2},\ a\in A,\quad \text{where } (a,2t)\in\Sigma A,$$

$$s(a,t)=(a,2t-1),\quad \tfrac{1}{2}\leq t\leqq 1,\ a\in A,\quad \text{where } (a,2t-1)\in W,$$

$$sx=x,\quad x\in V$$

[4] Of course, it was the *French* term "cofibration" which appeared, so that it is possible that it was not immediately recognized.

and we call this a *cooperation* of the cogroup ΣA on W. For any space Y, s induces a function

$$s^* : [W, Y] \times [\Sigma A, Y] \to [W, Y]$$

(since $[X_1 \vee X_2, Y] = [X_1, Y] \times [X_2, Y]$ in the category of based spaces), and s^* is an operation of the *group* $[\Sigma A, Y]$ on the based set $[W, Y]$. The paper was entitled "Operators and Cooperators in Homotopy Theory" and this bizarre title led the reviewer in *Mathematical Reviews,* Victor Gugenheim, to begin his review, "This is not a classification of homotopy-theorists but..."! It is worth noting that, in this paper, we did make our definitions in a general category (with zero maps). This move toward generality was to be maintained.

3.

Further papers on duality [8, 9; 27] in 1961 marked our movement toward a more explicit acceptance of Saunders MacLane's point of view that one should study duality in any category, and in particular in the category of groups. However, we combined this with the point of view of our earlier notes that one should, in particular, study groups (and cogroups) in any category. These approaches came together in a series of papers [10, 11, 12] on group-like structures in general categories. Here I would like only to refer to one result which, as it seems to me, is still far from being fully exploited.

We consider the category $\mathscr{G}$ of groups. If K is a group, then an H'-structure on K is a homomorphism $\mu : K \to K * K$, where $*$ denotes the free product, rendering commutative the diagram

$$\begin{tikzcd} K \arrow[r, "\mu"] \arrow[dr, "\Delta"'] & K * K \arrow[d, "\varrho"] \\ & K \times K \end{tikzcd} \tag{3.1}$$

where ϱ is the natural map from free to direct product and Δ is the diagonal. We then have

Proposition 3.1. *The group K admits an H'-structure if and only if it is free.*

Proof. It may readily be shown that, for *any* group K, the group $\varrho^{-1}(\Delta K)$ is free, generated by commutators $[x', x'']$, $x \in K$, $x \neq e$, the identity element[5]. Now if μ exists, then μ is injective and embeds K as a subgroup of $\varrho^{-1}(\Delta K)$. Thus K is free. Conversely, if K is free on the set $\{x_\alpha\}$ then an H'-structure on K is given by $\mu(x_\alpha) = x'_\alpha x''_\alpha$.

[5] Here we use x', x'' for the element x copied into the first, second factors of $K * K$, respectively.

Kan [30] made a very beautiful refinement of this result in 1958. He proved the following.

Proposition 3.2. *Let $\mu : K \to K * K$ be an associative H'-structure on K. Then the set of elements $\{x_\alpha\}$ in K, different from e, such that $\mu(x_\alpha) = x'_\alpha x''_\alpha$, is a free generating set for K.*

Sketch of proof. Since the μ-images of the x_α constitute an independent set, so do the x_α themselves. Thus it suffices to show that they generate K. Let $\bar{S}$ be the set of elements x such that $\mu x = x'x''$ (including e). Let $a \in K$ and let μa be written as $b'_1 c''_1 \cdots b'_n c''_n$, where only b_1, c_n are allowed to be e. The associativity of μ, expressed by the equation $(1 * \mu)\mu = (\mu * 1)\mu : K \to K * K * K$, implies that b_1, $c_n \in \bar{S}$; notice that, if for any element $u \in K$, $u = v'w''$, then the H'-axiom implies that $u = v = w$. We may now argue by induction on n that a belongs to the group generated by $\bar{S}$, and the proposition is proved.

The contribution of Eckmann and myself to this study of comultiplications is largely a matter of point of view. We observed that Kan's result essentially established a one-one correspondence between pointed sets $\bar{S}$ and associative H'-structures μ. For if $\bar{S}$ is a pointed set, let K be the free group on $S = \bar{S} - (e)$, and let $\mu : K \to K * K$ be the associative H'-structure, given by $\mu(x) = x'x''$, $x \in S$. Then Kan's theorem asserts that $\bar{S} \mapsto \mu$ defines a one-one correspondence. We now remark (a) such an associative H'-structure μ is actually a *cogroup* structure; and (b) the one-one correspondence $\bar{S} \mapsto \mu$ extends to a correspondence between pointed functions and cogroup-homomorphisms. To see (a) simply define $\sigma : K \to K$ by $\sigma x = x^{-1}$, $x \in S$; to see (b) let $\sigma : (K, \mu) \to (L, \nu)$ be a cogroup-homomorphism, so that

$$\begin{array}{ccc} K & \xrightarrow{\mu} & K * K \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi * \varphi} \\ L & \xrightarrow{\nu} & L * L \end{array} \tag{3.2}$$

is a commutative diagram of *group*-homomorphisms, and let $\bar{T} \mapsto \nu$. If $x \in \bar{S}$, then

$$\nu\varphi x = (\varphi * \varphi)\mu x = (\varphi * \varphi)(x'x'') = (\varphi x)'(\varphi x)'',$$

so that $\varphi x \in \bar{T}$. It is obvious, conversely, that any function $f : S \to T$ extends to a cogroup-homomorphism $\varphi : K \to L$. Thus we have established the following 'duality' between the category of groups $\mathcal{G}$ and the category of pointed sets $\mathcal{S}$.

Theorem 3.3. *The category $\mathcal{G}$ is the category of groups in $\mathcal{S}$; the category $\mathcal{S}$ is the category of cogroups in $\mathcal{G}$.*

In other words, just as we may base the notion of *group* on the notion of *set*, so may we base the notion of *set* on the notion of *group!*

We also see that both $\mathscr{G}$ and $\mathscr{S}$ show up as examples of *primitive categories,* in the sense of the second paper of the series referred to [11].

4.

The duality in homotopy theory, and its generalization in the sense of MacLane [33, 34] formed the subject of a considerable amount of further research. The duality itself was (at least in part) put on a functorial basis by the Russian topologists D.B. Fuks and A.S. Švarc [22, 36]. The work of Fuks led to important, purely category-theoretical, work on duality of functors and autonomous categories by Linton [32] and others. Articles by many authors discussed results which stood in evident duality and admitted dual proofs; dual results where the proofs were quite different; and dual statements only one of which was valid. Let us give one example of the second phenomenon.

Let X be a connected polyhedron. There is then a natural embedding $i : X \to \Omega\Sigma X$, adjoint to the identity on ΣX, and it may be shown that X is an H-space if and only if X embeds in $\Omega\Sigma X$ as a retract, that is, i has a left inverse. The best proof of this, to date, employs James' representation of $\Omega\Sigma X$ as the infinite reduced product X_∞ (the free monoid on X with base point as identity) [39]. Dually, there is a natural projection $p : \Sigma\Omega X \to X$, (co)adjoint to the identity on ΩX; and it may also be shown that X is an H'-space if and only if p has a right inverse. Here a proof may be given by modeling $\Sigma\Omega X$ as $E_1 X \cup_{\Omega X} E_2 X$, where $E_1 X$, $E_2 X$ are contractible path spaces on X. To this day, no dual proofs of these dual results are known.

If we confine attention to the homotopy theory of spaces, then there is, of course, in general no unique dual of a given concept, since that concept may not, in its original form, be expressed in categorical terms. A very interesting example of this is the concept of Lusternik–Schnirelmann category[6], about which James has recently written an important survey article [40]. This notion, developed to study critical points of smooth functions on manifolds, was originally defined in a very topological manner: the L–S category of X, cat X, is the minimum number of open sets, each contractible in X, which cover X. If X is a polyhedron, then, as shown by G.W. Whitehead, one may characterize the L–S category as follows. Let X^n be the n-fold cartesian power of X and let $T_n X$ be the subspace of X^n consisting of ordered n-tuples of points of X, at least one of which is the base point. Then cat $X \leq n$ if and only if the diagonal map $\Delta : X \to X^n$ may be compressed into $T_n X$. If one can give a categorical interpretation of $T_n X$ – and this is not difficult – then one can dualize the L–S category. This was the approach taken in [12] but it did not prove very fruitful in homotopy theory, because the dual concept was so intractable. The one success was that, with this dualization, we did have the result that cocat $X \leq 2$ if and

[6] The reader is warned that this use of the term "category" has nothing to do with the work of Eilenberg–MacLane!

only if X is an H-space, corresponding to the fact that cat $X \leq 2$ if and only if X is an H'-space.

It was Ganea who found what is probably the most useful dualization of L–S category [37][7]. His procedure is based on the characterizations of H-spaces and H'-spaces given above. Let us start with the embedding $o \rightarrowtail X$, which we call the 1-stage. If the n-stage is $F_nX \xrightarrow{f_n} X$, let $SF_nX \xrightarrow{g_n} F_nX$ embed the fibre of f_n in F_nX and let $F_nX \xrightarrow{h_n} F_{n+1}X$ project F_nX onto the cofibre of g_n. Then f_n factors canonically through h_n as $f_n = f_{n+1}h_n$, yielding the $(n+1)$-stage $F_{n+1}X \xrightarrow{f_{n+1}} X$. Ganea first showed that cat $X \leq n$ if and only if f_n admits a right inverse, generalizing the result that X is an H'-space if and only if $\Sigma\Omega X \rightarrow X$ admits a right inverse; it was then natural to dualize the procedure, starting with $X \rightarrow o$, to get a definition of cocat X. This definition has some very good properties: it yields a bound on the length of non-trivial iterated Whitehead products (just as the L–S category yields a bound on the length of non-trivial iterated cup products in cohomology); and, again dualizing a known result, if $p : E \rightarrow B$ is a fibre-map, with fibre F, then cocat $F \leq$ cocat $E + 1$. However it remains remarkably difficult to calculate cocat X – even more difficult than to calculate cat X, no inconsiderable task![8] For example, we do know that if S^n is an *odd*-dimensional sphere, then cocat $S^n = 2$, $n = 1, 3, 7$, since S^n is then an H-space; cocat $S^n = 3$ otherwise, since S^n is not then an H-space, but S^n is the fibre of a map

$$\Omega S^{n+1} \rightarrow \Omega S^{2n+1}.$$

However, the only *even*-dimensional sphere for which we know the cocategory[9] is S^2; cocat $S^2 = 3$, since S^2 is the fibre of the map, $BS^1 \rightarrow BS^3$, where BG is the classifying space for G. We do not even have a good understanding of when cocat X is finite.

The duality of which I have spoken is scarcely a topic of active research today, although, as we have indicated, there are still interesting open questions. Rather it is a commonplace of experience among topologists, accepted as "obvious". Every now and then, however, a new insight is provided into what were apparently totally familiar areas of homotopy theory by observing a new example of duality. Perhaps the most recent example is due to Peter May, who has noticed that the two famous Whitehead theorems,

[7] James wrote in [40] "The Whitehead definition of category can be dualized in an obvious way, but the result does not seem to be of much interest. However Ganea has proposed a different definition of cocategory which has desirable properties, for example the spaces of cocategory ≤ 2 are precisely the Hopf spaces."

This statement is misleading since (a) Ganea's definition of cocategory is a dualization of a characteristic property of category, and (b) the property quoted in favor of Ganea's definition is shared by the dualization of the Whitehead definition.

[8] We often need cohomological methods to get a lower bound on cat X, and analytical methods to get an upper bound.

[9] We may conjecture cocat $S^n \leq 4$.

(A) *a map* $f : X \to Y$ *between* 1*-connected spaces which induces homology isomorphisms is a weak homotopy equivalence;*

(B) *a map* $f : X \to Y$ *between connected CW-complexes which induces homotopy isomorphisms is a homotopy equivalence*

admit dual proofs. We should reinterpret (A) as asserting that f induces cohomology isomorphisms, and use homotopy decomposition (of Z) to show that f induces a bijection $f^* : [Y, Z_\infty] \to [X, Z_\infty]$, where Z_∞ is the inverse limit of the Postnikov tower of the 1-connected space Z. We prove (B) by using cellular decomposition (of Z) to show that f induces a bijection $f_* : [Z, X] \to [Z, Y]$, where Z is a connected CW-complex. Of course, (A) can be generalized to a map between nilpotent spaces.

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(The reader is referred to the Steenrod classification of reviews in homotopy theory for a more extensive bibliography).

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